EXACT SOLUTION OF THE HYPERBOLIC GENERALIZATION OF BURGERS EQUATION, DESCRIBING TRAVELLING FRONTS AND THEIR INTERACTION

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Abstract. We present new analytical solutions to the hyperbolic generalization of Burgers equation, describing interaction of the wave fronts. To obtain them, we employ a modified version of the Hirota method.

Keywords: Generalized Burgers equation, modified Hirota method, exact bi-soliton solutions.

Mathematics Subject Classification: 35C99, 34C60, 74J35.

1 Introduction

At the present time there are known only few analytical methods enabling to solve sufficiently general sets of initial or boundary value problems for nonlinear PDEs and they are applied to either completely integrable PDEs [1] or those PDEs that can be transformed to linear ones by means of non-local change of variables [2].

The majority of evolutionary PDEs used to simulate non-linear transport phenomena is not completely integrable. Yet it is well known that under certain conditions coherent structures formation take place during the transport phenomena occurring in open dissipative systems [3] and their analytical description is of great interest.

The simplest coherent structures are represented by periodic, quasiperoidic, kink-like and soliton-like travelling wave (TW) solutions. In recent decades a number of effective methods, enabling to obtain analytical expressions for TW solutions describing coherent structures have been put forward [4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. But most of the papers dealing with this subject concentrate upon the finding out solutions, evolving in a self-similar mode. And only few authors look for "bi-soliton" (or a "multi-soliton") solutions

to those PDEs, that are not completely integrable. Being informal, we mean by this term solutions describing interaction of TW regimes.

Perhaps, the most advanced study of bi-soliton solutions of PDEs that are not completely integrable has been performed by H. Cornille and A. Gervois [14, 15]. They described a broad class of non-linear PDEs, possessing bi-soliton solutions. Yet the methodology put forward by these authors applies merely to so called "factorized" PDEs and, besides, further investigations revealed that their classification is not complete. T. Kawahara and M. Tanaka considered equation of the Fisher type that does not fit the classification scheme from [15]. They succeeded in analytical description of travelling fronts, using the classical Hirota method [16].

In this work we present the analytical description of interacting wave fronts within the hyperbolic generalization of Burgers equation, using the modified Hirota method. The structure of the study is following. In section 2 we introduce a canonical form of generalized Burgers equation and study its TW solutions within the classical Hirota method. At the beginning of section 3, using the preliminary information about the TW regimes, we try to obtain bi-soliton solutions using the classical Hirota method [Dodd] and show that it is rather impossible. So in the following part of this section we use a modified version of the Hirota method, enabling to achieve our goal.

2 Hyperbolic generalization of Burgers equation and its TW solutions

We consider the hyperbolic generalization of Burgers equation [17]:

$$\tau u_{tt} + u_t + u\, u_x - \kappa\, u_{xx} = \sum_{\nu=0}^{3} a_\nu\, u^\nu. \tag{1}$$

Here $u = u(t, x)$, τ, κ, a_ν are constant parameters (for physical reason $\tau \geq 0$, and $\kappa > 0$). Lower indices denote partial derivatives with respect to corresponding variables. Assuming that the polynomial in RHS of equation (1) has three real roots, we can rewrite this equation in the following equivalent form:

$$\tau u_{tt} + u_t + u\, u_x + B\, u_x - \kappa\, u_{xx} = \lambda\, u\, (u - S)\, (u - Q)\,, \tag{2}$$

where B, S, Q, λ are constant parameters. We'll use for equation (2) the abbreviation GBE.

As it was previously announced, the main goal of this study is to obtain the analytical description to the bi-solitons, using the Hirota method. Employment of this method, that

will be briefly described later on, leads to a very complicated system of non-linear algebraic equations that rather could not be solved directly without any additional information about parameters being involved into the scheme. Processing of the system of algebraic equations appearing within the Hirota method becomes more simple if we restrict ourself to finding out solutions, possessing some already known asymptotic features. We will assume in this study that bi-soliton solutions tend on $+\infty$ or (and) $-\infty$ to corresponding TW solutions. For this reason we begin with the analytical description of kink-like TW solutions, playing the role of asymptotics. To obtain them, we use the ansatz

$$u = \frac{f_x}{f}. \tag{3}$$

Following the Hirota method, we put

$$f = 1 + \epsilon\, \varphi(t,\, x),$$

where $\varphi(t,\, x)$ is an unknown function and ϵ is a formally small parameter. We say that $\varphi(t,\, x)$ is "formally" small, since after the substitution of (3) into (2) we use the asymptotic decomposition of the expression obtained, treating this parameter as a small one, but finally we put $\epsilon = 1$. The detailed description of this procedure can be found e.g. in [1].

So, we insert ansatz (3) into (2) and multiply the obtained expression by f^3. This way a third-order polynomial with respect to ϵ is obtained. Equating to zero the coefficient of ϵ^1, we get the following PDE:

$$\tau\, \varphi_{ttx} + \varphi_{tx} + B\, \varphi_{xx} - \kappa\, \varphi_{xxx} + Q\, S\, \lambda\, \varphi_x = 0. \tag{4}$$

It is obvious, that solutions of the linear PDE (4) can be presented in the form

$$\varphi(t,\, x) = exp\left[a\, x - v\, t + c\right], \tag{5}$$

where $a,\ v,\ c$ are constant parameters. Inserting (5) into the ansatz (3), and equating to zero coefficients of $\epsilon^k,\ k = 1,\, 2,\, 3$, we obtain the following system of algebraic equations:

$$-\, a\, B + v + a^2\, \kappa - Q\, S\, \lambda - v^2\, \tau = 0, \tag{6}$$

$$-v + a^2(1+\kappa) + 2\, Q\, S\, \lambda + a\left[B + (S - Q)\, \lambda\right] - v^2 \tau = 0, \tag{7}$$

$$(a + Q)\, \lambda\, (a - S) = 0. \tag{8}$$

We see from equation (8) that in case when λ is non-zero parameter a is equal to either S or $-Q$.

In the first case we get the solution of the form

$$\varphi_1(t,\,x) = exp\left[S\,x - t\,v + c_1\right], \tag{9}$$

$$v = \frac{S\left[2\,B + S\,(\lambda+1) + 2\,\lambda\,Q\right]}{2}. \tag{10}$$

This function being inserted into (3) gives rise to a kink-like solution in case when an extra condition

$$\kappa = \kappa_v = \frac{\tau\left[2\,B + S + \lambda\,(2\,Q + S)\right]^2 - 2\,(1+\lambda)}{4} \tag{11}$$

is satisfied. As it was mentioned before, we finally put $\epsilon = 1$ in the formula (3). This may be done correctly because the parameter c in (5) is arbitrary. So we can "rescale" it as follows: $c \rightarrow c + log\,1/\epsilon$. This trick eliminates ϵ from the formula (3).

In the second case, i.e. when $a = -Q$ we get the following solution of system (6)–(7):

$$\varphi_2(t,\,x) = exp\left[-Q\,x - t\,w + c_2\right], \tag{12}$$

$$w = \frac{-Q\left[2\,B - Q\,(\lambda+1) - 2\,\lambda\,S\right]}{2}, \tag{13}$$

which leads to the kink-like TW solution providing that

$$\kappa = \kappa_w = \frac{\tau\left[\lambda\,(2\,S + Q) + Q - 2\,B\right]^2 - 2\,(1+\lambda)}{4}. \tag{14}$$

We see that, generally speaking, parameters κ_v and κ_w are different. But, in contrast to $a,\,v,\,w$ and $c_i,\ i = 1,\,2$, the parameters κ characterizes equation (2) and cannot be chosen arbitrarily. So in order that (9) and (12) describe two different solutions of GBE, we should equalize κ_v and κ_w. It is easy to show that equalities

$$\kappa_v = \kappa_w = \kappa = \frac{(Q+S)^2\,\tau\,(1 + 3\,\lambda) - 8\,(1+\lambda)}{16} \tag{15}$$

take place if

$$B = \frac{(S-Q)\,(\lambda-1)}{4}. \tag{16}$$

So when κ and B are given by equations (15) and (16) correspondingly, we get two different kink-like TW solutions of the same equation.

3 Bi-soliton solutions

3.1 Procedure based on the classical Hirota method

We look for the solution of the following form:

$$u(t,\,x) = F\left(\omega_1,\,\omega_2\right), \tag{17}$$

where $\omega_1 = S\,x - v\,t + c_1$, $\omega_2 = -Q\,x - w\,t + c_2$. Solution (17) describes interactions of various TW in cases when ω_1 and ω_2 are not proportional. As it was mention before, in our attempts to obtain bi-soliton solutions within the Hirota method, we face the necessity to solve very difficult systems of non-linear algebraic equations. To make the problem more tractable, we should pose some conditions on the parameters to be determined. So assuming that solution we are looking for describes asymptotically (depending on the sort of interaction) one or two TW fronts, we choose parameters v and w in accordance with the formulae (10) and (13). Of course, we can proceed this way further on and incorporate the formulae (15), (16) as well, but our analysis shows that in this case there is a lack of non-trivial solutions.

When looking for simplest bi-soliton solution, it is instructive to choose f in the form of superposition of functions corresponding to the TW solutions:

$$f + \epsilon\,(exp[\omega_1] + exp[\omega_2]) + R\,\epsilon^2\,exp[\omega_1 + \omega_2]. \tag{18}$$

Inserting (3) with f given by the formula (18) into equation (2) and multiplying the obtained expression by f^3, we get a six-order polynomial with respect to ϵ. Equating to zero coefficients of ϵ^n, $n = 1, 2, ...6$, we obtain six nonlinear algebraic equations (we denote them as (eq1),...(eq6)), containing the parameters and products of $exp[\omega_1]$ and $exp[\omega_2]$. For brevity we shall use the notation

$$x^m = exp[m \cdot \omega_1], \quad y^n = exp[n \cdot \omega_2].$$

Calculations were carried out in several steps essentially based upon employment of the "Mathematica" software. Below we describe a crucial points of the procedure employed, enabling to repeate it. Yet it is obvious, that correctness of solutions obtained within this procedure can be checked by direct inspection without going into details.

Before we proceed further on, let us note that we do not take into account solutions with $Q = 0$, $S = 0$, $Q = -S$ and $\lambda = -1/3$. Elementary analysis of formula (3) whth (10) and (13) being taken into account shows, that in all these cases we get at most TW solution.

So let us describe the procedure of obtaining bi-soliton solution based on the ansatz (3).

Step 1. First of all we consider (eq6), which is the simplest one:

$$Q\,R^3\,S\,(S - Q)\,x^3\,y^3\,\lambda = 0. \tag{19}$$

From this equation we obtain that

$$S = Q. \tag{20}$$

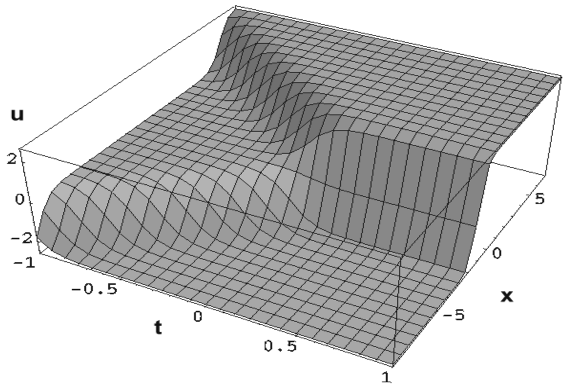

Figure 1: An example of a bi-soliton solution of equation (27), described by the formula (28) for $S = 2.5$, $c_1 = 2$, $c_2 = -6$, $\kappa = 0.5$

Step 2. Next we consider (eq1), which is a linear one with respect to variables $x,\ y$. Taking into account (20) and equating to zero coefficients of x and y, we obtain the following system:

$$-2\,(1+\lambda) - 4\,\kappa + \tau\left[4\,B^2 + 4\,B\,S\,(1+3\,\lambda) + S^2\left(1+6\,\lambda+9\,\lambda^2\right)\right] = 0, \tag{21}$$

$$2\,(1+\lambda) + 4\,\kappa + \tau\left[-4B^2 + 4\,B\,S\,(1+3\,\lambda) - S^2\left(1+6\,\lambda+9\,\lambda^2\right)\right] = 0. \tag{22}$$

Summing up these two equations we get the equation

$$8\,B\,S\,\tau\,(1+3\,\lambda) = 0. \tag{23}$$

To fulfill (23), we put $B = 0$. Returning then to equation (21) we obtain the following expression for κ:

$$\kappa = \frac{-2\,(1+\lambda) + \tau\,S^2\,(1+6\,\lambda+9\,\lambda^2)}{4}.$$

Step 3. With these conditions (eq1), (eq2) and (eq6) nullify while subsystem (eq3)–(eq5) becomes as follows:

$$S^3\,x\,y\,(x-y)\,(1+3\,\lambda)\left[2\,S^2\,\tau\,(1-R)\,(1+3\,\lambda) + R\right] = 0, \tag{24}$$

$$R\,S^3\,x\,y\left(x^2-y^2\right)(1+3\,\lambda) = 0, \tag{25}$$

$$R^2\,S^3\,x^2\,y^2\,(x-y)\,(1+3\,\lambda) = 0. \tag{26}$$

The only non-trivial solution to (24)-(26) corresponding to bi-solitons is $R = \tau = 0$.

It can be easily checked starting from equation (23) that we do not obtain any other solution choosing solution $\tau = 0$ instead of $B = 0$ in the second step.

Thus, we became convinced that there is impossible to obtain a bi-soliton solution satisfying (2), using the classical Hirota method. Yet as a by-product we get the following result:

Theorem 1. *The Burgers equation*

$$u_t + u\,u_x - \kappa\,u_{xx} = -\left(1 + 2\,\kappa\right)\,u\,\left(u^2 - S^2\right) \tag{27}$$

possess a bi-soliton solution

$$u(t,\,x) = S\,\frac{exp\left(S\,x - v\,t + c_1\right) - exp\left(-S\,x - v\,t + c_2\right)}{1 + exp\left(S\,x - v\,t + c_1\right) + exp\left(-S\,x - v\,t + c_2\right)} \tag{28}$$

where $v = -S^2\left(1 + 3\,\kappa\right)$.

Example of bi-soliton solution describing interaction of two wave fronts is shown in Fig. 1.

3.2 Procedure based on the modified Hirota method

One can succeed in obtaining bi-soliton solutions to (2) by slightly modifying Hirota method. Let us consider the following ansatz:

$$u(t,\,x) = \frac{g}{f}, \tag{29}$$

in which we put

$$g = \epsilon\left[\alpha\,exp\left(\omega_1\right) + \beta\,exp\left(\omega_2\right)\right] + \epsilon^2\,A\,exp\left(\omega_1 + \omega_2\right), \tag{30}$$

$$f = 1 + \epsilon\left[exp\left(\omega_1\right) + exp\left(\omega_2\right)\right] + \epsilon^2\,R\,exp\left(\omega_1 + \omega_2\right). \tag{31}$$

So we insert (29) into (2), multiply the resulting equation by f^3 and obtain this way a six-order polynomial with respect to ϵ. Equating to zero coefficients of ϵ^k, $k = 1, 2, ...6$, we get a system of non-linear algebraic equations which we again denote by $(eq1)$–$(eq6)$. Using the "Mathematica" software we succeeded in obtaining a non-trivial solustion of these equations. A sketch of the procedure employed is presented below.

Step 1. We consider (eq6) which reads as follows:

$$-A\,\left(A + Q\,R\right)\left(A - S\,R\right)x^3 y^3 = 0.$$

In order to satisfy this equation, we put $A = R\,S$.

Step 2. Next we consider (eq1). Nullifying the coefficient of y^1 we obtain the equation

$$Q^2\,\beta\,\Big\{-2\,(1+2\,\kappa)+\tau\left(4\,B^2-4\,B\,Q+Q^2\right)+(Q+2\,S)^2\,\lambda^2\tau+ \\ +\lambda\left[-2\left(1-Q^2\tau\right)+4\,Q\,S\,\tau-4\,B\,\tau\,(Q+2\,S)\right]\Big\}=0.$$

This equation will be satisfied if we choose $\beta=0$. With this choice we consider (eq5) and, nullifying coefficient of $x^3\,y^2$ obtain:

$$(S-\alpha)\,\Big[4\,Q\,S-4\,S^2\lambda+\tau\,Q^4\,(1+\lambda)^2+4\,Q^3\tau\,(1+\lambda)\,(S\,\lambda-B)+ \\ +2\,Q^2\left(1-2\,\kappa+\lambda+2\,B^2\tau-4\,B\,S\,\lambda\,\tau+2\,S^2\lambda^2\tau\right)\Big]=0.$$

This equation will be satisfied if $S=\alpha$. Equating to zero the remaining term, we get:

$$-\frac{1}{4}R^2S^3x^2y^3\,\Big\{-2-4\,\kappa+4\,B^2\tau+4\,B\,S\,\tau+S^2\tau+(2\,Q+S)^2\,\lambda^2\tau+ \\ +2\,\lambda\left[-1+2\,Q\,S\,\tau+S^2\tau+2\,B\,(2\,Q+S)\,\tau\right]\Big\}=0. \tag{32}$$

Equation (32) is satisfied if κ is expressed by the formula (11).

Step 3. Next we consider (eq3). The coefficient of x^2y reads as follows:

$$-\frac{1}{4}\,Q\,(R-1)\,S\,(Q+S)^2\,(1+3\,\lambda)\,\tau\,(4\,B+2\,S-Q+\lambda\,Q+2\,\lambda\,S)$$

In order to nullify it we put

$$B=-\frac{2\,S-Q+\lambda\,(Q+2\,S)}{4}. \tag{33}$$

The remaining coefficient of (eq3) to be nullified is that of $x\,y^2$. Taking (33) into account we can write it as follows:

$$-Q\,(R-1)\,S\,(Q+S+Q\,\lambda+2\,S\,\lambda)\,.$$

Analysis of formula (29) shows that option $R=1$ leads to a one-soliton solution so we nullify this expression assuming that

$$\lambda=-\frac{Q+S}{Q+2\,S}. \tag{34}$$

Step 4. Taking into account all the above conditions and repeating the procedure we conclude that only (eq2) and (eq4) are nonzero: (eq2) is given by the formula

$$\frac{Q^2\,(R-1)\,S^2x\,y\,[2\,Q+4\,S+4\,Q^2\tau+8\,Q^2S\,\tau+5\,Q\,S^2\tau+\tau\,S^3]}{2\,(Q+2\,S)^2},$$

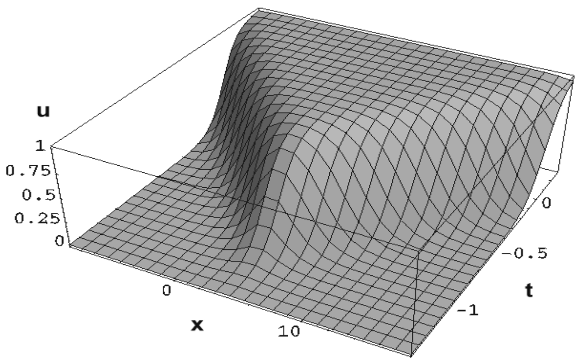

Figure 2: Creation of two wave fronts "from nothing", described by the formula (36) for $S = 1,\ Q = -1.9, c_1 = 10,\ c_2 = -6,\ R = 0$

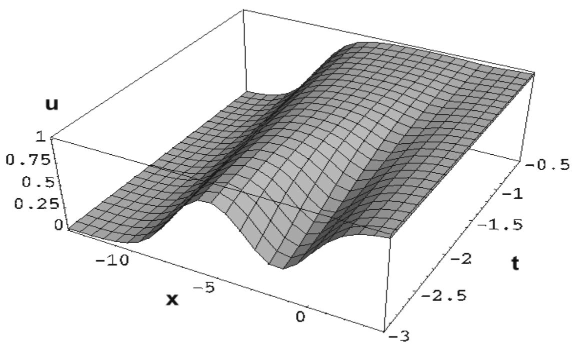

Figure 3: Interaction of localized TW solution and kink described by the formula (36) for $S = 1,\ Q = 2,\ c_1 = 2,\ c_2 = -3,\ R = 1500$

while (eq4) is as follows:

$$\frac{R\,Q^2\,(R-1)\,S^2x^2\,y^2\,[2\,Q + 4\,S + 4\,Q^2\tau + 8\,Q^2S\,\tau + 5\,Q\,S^2\tau + \tau\,S^3]}{2\,(Q+2\,S)^2}.$$

Both of them are nullified if

$$\tau = -\frac{2\,(Q+2\,S)}{(Q+S)\,(2\,Q+S)^2}. \tag{35}$$

And this way we obtained the bi-soliton solution to GBE. Elementary but slightly tiresome analysis shows that system (GBE) does not admit any other bi-soliton solution described by the formula (29). The only exception is the solution connected with the already obtained solution by means of the transformation $S \mapsto Q,\ Q \mapsto -S,\ \alpha = 0,\ \beta = -Q$. But existence of the second solution is a mere consequence of the arbitrariness of the choice of what is ω_1 and what is ω_2 in the formulae (30)–(31). It is obvious then, that this way we don't get any other independent solution.

Let us summarize the result obtained as the following statement.

Theorem 2. *If*

$$B = \frac{2\,Q - S}{4}, \quad \lambda = -\frac{Q+S}{Q+2\,S}, \quad \tau = -\frac{2\,(Q+2\,S)}{(Q+S)(2\,Q+S)^2}$$

and

$$\kappa = -\frac{Q+2\,S}{8\,(Q+S)}$$

then equation (2) admits the following bi-soliton solution

$$u(t,\,x) = \frac{S\,exp\,(\omega_1)\,[1+R\,exp\,(\omega_2)]}{1+exp\,(\omega_1)+exp\,(\omega_2)+R\,exp\,(\omega_1+\omega_2)}, \tag{36}$$

where

$$\omega_1 = S\left[x + \frac{t}{4}\,\frac{Q\,(2\,Q+S)}{Q+2\,S}\right] + c_1, \quad \omega_2 = -Q\left[x - \frac{t}{4}\,(2\,Q+S)\right] + c_2,$$

$S,\ Q,\ R,\ c_1,\ c_2$ *are arbitrary constants.*

Patterns of the interacting wave fronts described by the formula (36) are shown in Figs. 2, 3.

4 Concluding Remarks

Thus, we have obtained bi-soliton solutions for classical Burgers equation with cubic non-linearity and for the GBE. To our knowledge, such solutions for the equation (2) have been obtained for the first time. In this study we looked for the simplest bi-soliton solutions described by the formula (29). It is obvious that rational solutions of this sort can be unlimitedly generalized. However it should be noted, that one faces very non-trivial technical problems when incorporating additional terms into (30)–(31). And for successful application of the above method it is desired to have some extra information about the possible values of the parameters. Beside the asymptotic analysis, one can incorporate another methods, such as the Painlevé analysis [18] and study of conserved quantities [19].

Let us mention in conclusion that solutions obtained in this work describe merely interaction of kinks. It would be interesting to obtain an analytical description of interacting TW waves of another sort. In paper [20] there were obtained periodic, soliton-like and some other solutions to GBE. All of them have been obtained within the methods different from that used in this paper. Consequently a successful investigations of the interactions of TW of different types should be based on employment of more sophisticated ansatzes.

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